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Thermionic versus radiative cooling.

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1 Introduction

This paper is to answer doubts of Kay Wittenburg [1] about the left plot of Figure 5 in paper [2], where it is shown that at high temperature the thermionic cooling is more effective than the radiative one.

A short calculation is made and documented as the problem is not trivial. The calculation is mixed: numerical and analytical. Hopefully it shows clearly and in a convincing manner that the thermionic cooling becomes more powerful than the radiative one at temperatures about 3000 K.

In addition, Section 3 of this document explains why the thermionic cooling term contains component $2\mathrm{k_B T}$.

2 When thermionic emission dominates over radiation cooling

The title of this chapter can be expressed by Equation 1, where the symbols are explained in Table 1 of [2]. The work function ϕ is expressed in Volts (ϕ_V) or in Joules (ϕ_J). Numerically: $\phi_\mathrm{J} = \mathrm{q_e}\phi_\mathrm{V}$.

$$\mathrm{A_{rad}}(\phi_\mathrm{V} + \frac{2\mathrm{k_B T}}{\mathrm{q_e}})\mathrm{A_R T^2}\exp(-\frac{\phi_\mathrm{J}}{\mathrm{k_B T}}) > \mathrm{A_{rad}}\epsilon\sigma(\mathrm{T}^4 - \mathrm{T}^4_\mathrm{amb}) \quad (1)$$

Simplifying this inequality and looking for the critical temperature ($\mathrm{T_c}$) at which both processes contribute equally to the cooling:

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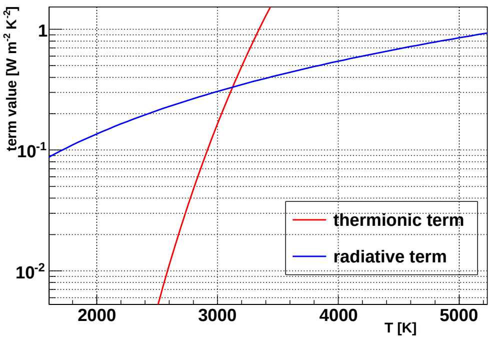

Figure 1: *Visualization of the left (red) and right (blue) term of Equation 3 for* $\phi_V = 4.5$ V.

$$(\phi_V + \frac{2k_BT_c}{q_e})A_RT_c^2\exp(-\frac{\phi_J}{k_BT_c}) = \epsilon\sigma(T_c^4 - T_{amb}^4) \tag{2}$$

As $T_c >> T_{amb}$, the Equation can be further simplified:

$$(\phi_V + \frac{2k_BT_c}{q_e})A_R\exp(-\frac{\phi_J}{k_BT_c}) = \epsilon\sigma T_c^2 \tag{3}$$

The typical values of the work function ϕ_V for carbon are between 4 and 5 V. The $\frac{2k_BT_c}{q_e}$ term, for 3100 K is 0.53 V, which is only 10% of the work function.

No analytical solution of the Equation 3 has been found. The numerical solutions for both terms, for $\phi_V = 4.5$ V, are presented graphically in Figure 1. It can be seen that the thermionic cooling becomes more important than the radiation cooling for $T_C \approx 3100$ K.

As the functions describing the cooling processes are complex, it should be proven that the thermionic term is larger than radiative one for temperatures above the numerical solution T_C. Writing the ratio of both terms and taking only the part of the thermionic term which contains the work function in front of the exponent (which is leading at T of about 3000 K) we get:

$$\frac{\phi_V A_R\exp(-\frac{\phi_J}{k_BT})}{\epsilon\sigma T^2} \tag{4}$$

By using the substitution $x = \frac{k_BT}{\phi_J}$, and for simplicity taking $\epsilon = 1$ (perfect black body) and using the fact that $\phi_J = \phi_V q_e$, the above expression can be transformed to:

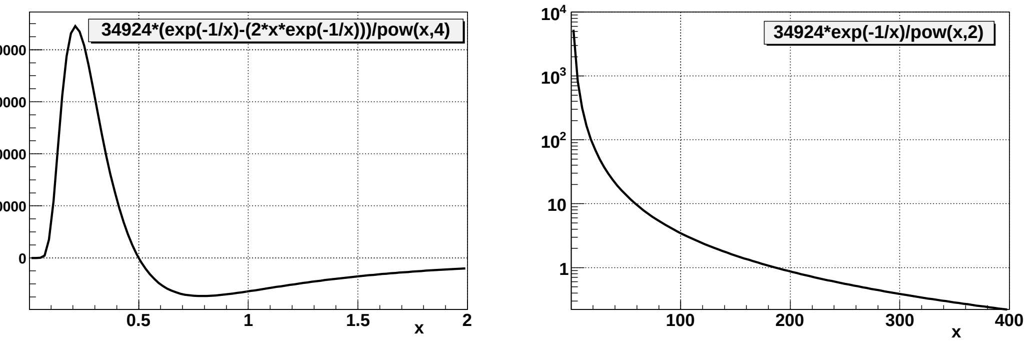

Figure 2: *Left plot: derivative of the ratio of thermionic and radiative terms. Right plot: ratio of thermionic and radiative term for very high temperatures.*

$$\frac{\mathrm{k_B^2 A_R}}{\sigma \mathrm{q_e^2} \phi_\mathrm{V}} \frac{\exp(-\frac{1}{\mathrm{x}})}{\mathrm{x}^2} = 34942.1 \frac{\exp(-\frac{1}{\mathrm{x}})}{\mathrm{x}^2} \tag{5}$$

For T=3100 K the variable x=0.0593 and the value of the whole expression is 0.4758. It becomes greater than one for x $\approx$ 0.0625 i.e. for T $\approx$ 3265 K.

As can be seen in Figure 1 in the thermionic term raises much faster than radiative one for 2000 K $<$ T $<$ 5000 K. The rate of the slopes can be estimated by differentiation of Equation 6:

$$34942.1 \frac{\mathrm{d}}{\mathrm{dx}} \frac{\exp(-\frac{1}{\mathrm{x}})}{\mathrm{x}^2} = 34924.1 \frac{\exp(-\frac{1}{\mathrm{x}}) - 2\mathrm{x} \exp(-\frac{1}{\mathrm{x}})}{\mathrm{x}^4} \tag{6}$$

This derivative is equal zero for x=0.5 (T = 26122 K), what means that up to this temperature the thermionic term raises faster than the radiative one. Of course for such a temperature the carbon fiber does not exist anymore.

The behaviour of the derivative is shown on the left plot of Figure 2. The right plot of this Figure has been prepared to fill the purely academic interest in the value of temperature at which the radiative term dominates over thermionic again, i.e. when ratio 6 is smaller than 1. This takes place for x $\approx$ 180 i.e. for T $\approx$ 9 million K, close to the estimated temperature in the core of the Sun (13 million K).

A similar calculation can be performed using the second contribution to the thermionic term (substituting $\epsilon = 1$):

$$\frac{2\mathrm{k_B T A_R} \exp(-\frac{\phi_\mathrm{J}}{\mathrm{k_B T}})}{\mathrm{q_e} \epsilon \sigma \mathrm{T}^2} = \frac{2\mathrm{k_B A_R}}{\mathrm{q_e} \sigma} \frac{\exp(-\frac{\phi_\mathrm{J}}{\mathrm{k_B T}})}{\mathrm{T}} \tag{7}$$

Using the same variable substitution as for the previous calculation it can be written:

$$\frac{2\mathrm{k_B A_R}}{\mathrm{q_e}\sigma}\frac{\exp(-\frac{\phi_\mathrm{J}}{\mathrm{k_B T}})}{\mathrm{T}} = \frac{2\mathrm{k_B^2 A_R}}{\mathrm{q_e^2}\sigma\phi_\mathrm{V}}\frac{\exp(-\frac{1}{\mathrm{x}})}{\mathrm{x}} = 69884.2\frac{\exp(-\frac{1}{\mathrm{x}})}{\mathrm{x}} \tag{8}$$

This expression, for the approximate value of the solution $\mathrm{T} = 3100$ K, is equal to about 0.0559 and it reaches 1 for $\mathrm{x} = 0.0725$, i.e. for temperature $\mathrm{T} \approx 3788$ K.

The derivative of the Expression 8 is:

$$69884.2\frac{\mathrm{d}}{\mathrm{dx}}\frac{\exp(-\frac{1}{\mathrm{x}})}{\mathrm{x}} = \frac{69884.2}{\mathrm{x}^3}\exp(-\frac{1}{\mathrm{x}})(1-\mathrm{x}) \tag{9}$$

It means that the thermionic term raises faster than the readiative one up to temperature of 52244 K.

The above considerations prove that the thermionic cooling is stronger than the radiative one for temperatures larger than $\mathrm{T_c}$ (which is a little bit more than 3100 K for $\phi_\mathrm{V} = 4.5$ V) and within the temperature range where the physical model is applicable.

3 Why thermionic emission term contains $2\mathrm{k_B T}$?

Another question which appears often in the discussions about thermionic cooling concerns the amount of energy removed from the wire by a single emitted electron. This energy is a sum of work function and thermal component $2\mathrm{k_B T}$. The question is why the thermal component is not $\frac{3}{2}\mathrm{k_B T}$, as for ideal gas? The calculation uses classical, Maxwell-Boltzmann distribution. The population of states of the electrons emitted from cathode is:

$$\mathrm{dN} = \mathrm{A}\exp(-\frac{\mathrm{E}-\mathrm{E_0}}{\mathrm{k_B T}})\mathrm{dE} \tag{10}$$

where $\mathrm{E_0}$ is surface potential energy. The electrons emitted from the surface (in the direction z, perpendicular to the surface), have the following average value of energy:

$$\bar{\mathrm{E}}_\mathrm{z} = \int_{\mathrm{E_0}}^{\infty}\mathrm{E_z dN_z} / \int_{\mathrm{E_0}}^{\infty}\mathrm{dN_z} \tag{11}$$

Solving the integrals by contributing $\epsilon_\mathrm{z} = \mathrm{E_z} - \mathrm{E_0}$:

$$\begin{aligned}\bar{\mathrm{E}}_\mathrm{z} &= \int_0^\infty(\epsilon_\mathrm{z}+\mathrm{E_0})\exp(-\frac{\epsilon_\mathrm{z}}{\mathrm{k_B T}})\mathrm{d}\epsilon_\mathrm{z} / \int_0^\infty\exp(-\frac{\epsilon_\mathrm{z}}{\mathrm{k_B T}})\mathrm{d}\epsilon_\mathrm{z} \\ &= \mathrm{E_0} + \int_0^\infty\epsilon_\mathrm{z}\exp(-\frac{\epsilon_\mathrm{z}}{\mathrm{k_B T}})\mathrm{d}\epsilon_\mathrm{z} / \int_0^\infty\exp(-\frac{\epsilon_\mathrm{z}}{\mathrm{k_B T}})\mathrm{d}\epsilon_\mathrm{z}\end{aligned}$$

Substituting $\mathrm{z} = -\frac{\epsilon_\mathrm{z}}{\mathrm{k_B T}}$ we get:

$$\bar{E}_z = E_0 - kT \int_0^{-\infty} z\,\exp(z)dz / \int_0^{-\infty} \exp(z)dz$$
$$= E_0 + kT$$

The total average energy must take into account the Fermi energy level E_F and the energy of the two degrees of freedom in direction parallel to the surface of the cathode: $2 \times \frac{1}{2} k_B T$:

$$\bar{E} = E_0 - E_F + k_B T + k_B T = \phi + 2k_B T \tag{12}$$

And this is how the thermionic cooling term is calculated.

4 Conclusion

This paper discusses the power of the two high-temperature cooling mechanisms: radiative cooling and thermionic emission. It shows that at temperatures above about 3100 K, the thermionic cooling is much more efficient than the radiative one. This statement is valid in all range of temperatures when there still exist a solid carbon.

The purely analytical, and therefore elegant solution to the problem has not been found. Probably it could be done by a student, who is freshly after mathematical analysis courses. Nevertheless author enjoyed a lot doing these small calculations.

References

[1] Kay Wittenburg, private communication at CAS 2009, Darmstadt.

[2] M. Sapinski, " Model of carbon wire heating in accelerator beam", CERN-AB-2008-030-BI report